Perfect Diffusion Primitives for Block Ciphers
–
Building Efficient MDS Matrices

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Outline of this talk

- Preliminaries
- MDS Matrices ...
- ... and their Implementation
- Bi-Regular Arrays
- Some New Constructions

Back to Shannon

- Notions of *confusion* and *diffusion* introduced by Shannon in *"Communication Theory of Secrecy Systems" (1949)*
- Confusion: *"The method of confusion is to make the relation between the simple statistics of $E_K(.)$ and the simple description of K a very complex and involved one."*
- Diffusion: *"In the method of diffusion the statistical structure of M which leads to its redudancy is dissipated into long range statistics – i.e., into statistical structure involving long combinations of letters in the cryptogram."*

Confusion

- Notion of confusion nowadays related to the ones of
  - S-Box
  - non-linearity
  - Boolean functions
  - algebraic attacks
- Plenty of academic papers on this subject !

Diffusion : Historical Perspectives

- Less studied in a rigorous (mathematical) way until mid of 90's
- Schnorr-Vaudenay (FSE'93 / EUROCRYPT'94) : introduction of the concept of *multipermutation*
- Vaudenay (FSE'95) : a *linear* multipermutation is equivalent to an MDS code
- Daemen (PhD thesis, 1995): *Wide-Trail Strategy*
  - (Choose "good" S-boxes)
  - "*Design the round transformation in such a way that only trails with many S-boxes occur*."
- Rijmen, Daemen, Preneel, Bossalaers, De Win (FSE'96): design of SHARK whose diffusion layer is based on MDS codes

Multipermutation Nowadays

- Very few MDS codes are known
- Seldom used in practice
- Widely spread building block in symmetric schemes
- Non-linear multipermutation: CS-Cipher
- Linear multipermutation (MDS matrices): AES, Camellia, Twofish, Khazad, FOX, and many, many others !

In this Talk

- Interested in “efficient” linear multipermutations
- Brief recall about MDS matrices and their properties
- Definition of what we mean by “efficient”
- New propositions

Multipermutation: a Definition

Definition (Multipermutation)

A diffusion function f from $\mathcal{K}^p$ to $\mathcal{K}^q$ is a *multipermutation* if for any $x_1, \ldots, x_p \in \mathcal{K}$ and any integer r with $1 \leq r \leq p$, modifying r input values on $f(x_1, \ldots, x_p)$ results in modifying at least $q - r + 1$ output values.

Multipermutation: Another Definition

- **Definition (Multipermutation)**
  A diffusion function f from $\mathcal{K}^p$ to $\mathcal{K}^q$ is a *multipermutation* if the set of all words consisting of $x_1, \ldots, x_p$ concatenated with $f(x_1, \ldots, x_p)$ is a code of $(\#\mathcal{K})^p$ words of length $p + q$ with minimal distance $q + 1$.
- Matches the Singleton bound (hence the link to MDS codes)

Multipermutation: Example

- Representation of the finite field $\mathrm{GF}(2^8)$: polynomials of degree at most seven with coefficients in $\mathrm{GF}(2)$ modulo the irreducible polynomial

$$p(\xi) = \xi^8 + \xi^7 + \xi^6 + \xi^5 + \xi^4 + \xi^3 + 1$$

  - Addition: XOR
  - Multiplication: usual multiplication of polynomials modulo $p(\xi)$
- Consider the following multipermutation on $\mathrm{GF}(2^8)^2$:

$$\mu : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & \xi \\ 1 & 1 \end{pmatrix} \times \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Why is it a Multipermutation ?

- Because μ is invertible :

$$\begin{pmatrix} 1 & \xi \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \xi^7 + \xi^5 + \xi^3 & \xi^7 + \xi^5 + \xi^3 + 1 \\ \xi^7 + \xi^5 + \xi^3 & \xi^7 + \xi^5 + \xi^3 \end{pmatrix}$$

- Because, when fixing x_1 to a constant c, *both* y_1 and y_2 are permutations of x_2 :

$$\begin{aligned} y_1 &= c \oplus (\xi \cdot x_2) \\ y_2 &= c \oplus x_2 \end{aligned}$$

- Because, when fixing x_2 to a constant c, *both* y_1 and y_2 are permutations of x_1 :

$$\begin{aligned} y_1 &= x_1 \oplus (\xi \cdot c) \\ y_2 &= x_1 \oplus c \end{aligned}$$

Why is it a Multipermutation (2)?

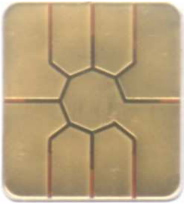
- Because $\det(\mu) \neq 0$ and every sub-determinant of μ is different of 0.

32/64-bit Architectures

- Lot of fast memory (L1 cache)
- Table lookups + XORs:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = x_1 \times \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \oplus \quad x_2 \times \begin{pmatrix} \xi \\ 1 \end{pmatrix}$$

8-bit Architectures



- Less memory at disposal $\rightarrow$ complete precomputation is impossible!
- The matrix elements value matters !
- Multiplications by 1 are “free” operations
- Possible to precompute the operation “multiplication by a constant c”

Our strategy

- Maximize the number of 1's in the matrix.
- Minimize the number of different constants.
- Two criteria ...
- ... among infinitely many others !
- Corollary (and disclaimer) : it is *always* possible to find an architecture and side constraints such that our strategy leads to poor results.
- One of the constraints we did **not** consider: inverse of a matrix must be "efficient" as well.

Results

- Definition of the concept of “bi-regular array”
- Find the minimal amounts of 1’s and of different coefficients for bi-regular arrays
- Sequence of constructive proofs $\longrightarrow$ matrix skeletons
- Examples of matrices

Bi-Regular Arrays

- A 2×2 array with entries in $\mathcal{K}$ is *bi-regular* if at least one row **and** one column have two different entries.

1	1
1	2

1	1
ξ	ξ

- A $q \times p$ array with entries in $\mathcal{K}$ is *bi-regular* if all 2×2 sub-arrays are bi-regular.
- An MDS matrix must be a bi-regular array ...
- ... but the converse is not true !

From Bi-Regular Arrays to MDS Matrices

- Construct a bi-regular array with large number of 1's and small number of different coefficients.
- Find a suitable set of coefficients (if possible).

Highest Possible Number of 1’s

Summary of our results

	2	3	4	5	6	7	8
2	3	4	5	6	7	8	9
3	4	6	7	8	9	10	11
4	5	7	9	10	12	13	14
5	6	8	10	12	13	14	17
6	7	9	12	13	16	18	19
7	8	10	13	14	18	21	22
8	9	11	14	17	19	22	24

Lowest Possible Number of Different Coefficients

Summary of our results

	2	3	4	5	6	7	8
2	2	2	2	3	3	3	3
3	2	2	3	3	3	3	2
4	2	3	3	3	4	4	4
5	3	3	3	3	4	4	4
6	3	3	4	4	4	4	5
7	3	3	4	4	4	4	5
8	3	4	4	4	5	5	5

A $(4,4)$-Multipermutation

- Example of "optimal" 4×4-matrix

$$\begin{pmatrix} a & 1 & 1 & 1 \\ 1 & 1 & b & a \\ 1 & a & 1 & b \\ 1 & b & a & 1 \end{pmatrix}$$

- 9 coefficients equal to 1, 3 different values
- Used as diffusive component in the round function of FOX64

A Circulating-Like $(8, 8)$-Multipermutation

- Example of a "non-optimal" 4×4-matrix

$$\begin{pmatrix} f & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & a & b & c & d & e & f \\ 1 & f & 1 & a & b & c & d & e \\ 1 & e & f & 1 & a & b & c & d \\ 1 & d & e & f & 1 & a & b & c \\ 1 & c & d & e & f & 1 & a & b \\ 1 & b & c & d & e & f & 1 & a \\ 1 & a & b & c & d & e & f & 1 \end{pmatrix}$$

- Used as diffusive component in the round function of FOX128

A $(8,8)$-Multipermutation with Rectangle Patterns

- Example of a "partially optimal" 8×8-matrix

$$\begin{pmatrix} b & a & c & b & d & c & 1 & d \\ b & c & a & d & b & 1 & c & 1 \\ c & b & d & a & 1 & b & 1 & c \\ c & d & b & 1 & a & 1 & b & d \\ d & c & 1 & b & 1 & a & d & b \\ d & 1 & c & 1 & b & d & a & c \\ 1 & d & 1 & c & d & b & c & a \\ 1 & 1 & d & d & c & c & b & b \end{pmatrix}$$

- Optimal number of different coefficients
- Non-optimal number of 1's

Thank You !

See you in 25 minutes for the presentation of

Any Question ?